Math 246C Lecture 9 Notes

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1 Polya's Theorem and Universal Covering Spaces

1.1 Polya's theorem (cont.)

Last time, we were proving Polya's theorem. Let's finish the proof.

Theorem 1.1 (Polya, Ehrenpreis, Martineau)**.** *Let $K \subseteq \mathbb{C}$ be compact and convex, and let $\mathcal{M} \in \mathrm{Hol}(\mathbb{C})$ be such that*

$$|\mathcal{M}(\zeta)| \leq C_\delta \exp(H_K(\zeta) + \delta|\zeta|).$$

Then there exists a unique analytic functional μ such that $\widehat{\mu} = \mathcal{M}$ and μ is carried by K.

Proof. Set

$$\mu(f) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) f B \, d\lambda(\zeta),$$

where $\chi \in C_0^\infty(\mathbb{C})$ is 1 on a large disc and B is the Borel transform of $\mathcal{M}$. We claim that B can be extended analytically to $\hat{\mathbb{C}} \setminus K$. First, if the claim holds, μ is carried by K: for any neighborhood ω of K, we can choose $\chi \in C_0^\infty(\omega)$ such that $\chi = 1$ in a neighborhood of K.

Proof of claim: Let $w \in \mathbb{C}$ with $|w| = 1$, and let

$$B_w(\zeta) = \int_0^\infty \mathcal{M}(tw) w e^{-tw\zeta} \, dt.$$

We have

$$|\mathcal{M}(tw)e^{-tw\zeta}| \leq C_\delta \exp(tH_K(w) + \delta t - t\,\mathrm{Re}(w\zeta)).$$

Let $\Pi_w = \{\zeta \in \mathbb{C} : \mathrm{Re}(w\zeta) > H_K(w)\}$. It follows that $B_w \in \mathrm{Hol}(\Pi_w)$. When $\zeta \in \mathbb{C}$ is such that $w\zeta$ is real and $\gg 0$, then we can compute $B_w(\zeta)$ by expanding $\mathcal{M}(tw) = \sum_{j=0}^\infty \mathcal{M}^{(j)}(0)(tw)^j/j!$ as a Taylor series and integrating term by term. In general, if $f \in \mathrm{Hol}(|z| < R)$ and $|f| \leq M$, then Cauchy's estimates give

$$\left| f(z) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j} z^j \right| \leq \sum_{j=n}^\infty \frac{|f^{(j)}(0)|}{j!} |z|^j \leq M \left(\frac{|z|}{R}\right)^n \frac{1}{1 - |z|/R},$$

so integrating the Taylor series term by term is justified.

We get

$$B_w(\zeta) = \sum_{j=0}^{\infty} \frac{\mathcal{M}^{(j)}(0)}{j!} w^{j+1} \underbrace{\int_0^\infty t^j e^{-tw\zeta}\, dt}_{=j!(w\zeta)^{-(j+1)}} = B(\zeta)$$

for any w. It follows that for any w_1, w_2, B_{w_1}, B_{w_2} coincide in the region $\Pi_{w_1} \cap \Pi_{w_2}$, for they are both equal to B far away. We get a well-defined holomorphic function on $\bigcup_{|w|=1} \Pi_w$ which analytically continues B. Now

$$\bigcup_{|w|=1} \Pi_w = \{\zeta \in \mathbb{C} : H_k(w) < \mathrm{Re}(w\zeta) \text{ for some } w\} = \mathbb{C} \setminus K,$$

as we checked that $K = \{\zeta : \mathrm{Re}(z\zeta) \leq H_K(z)\, \forall z \in \mathbb{C}\}$. □

Remark 1.1. Let μ be an analytic functional. Then there is a compact set $K \subseteq \mathbb{C}$ and a measure ν on K such that

$$\mu(f) = \int_K f(z)\, d\nu(z).$$

By Cauchy's integral formula,

$$f(z) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \frac{f(\zeta)}{\zeta - z}\, d\lambda(s), \qquad z \in K,$$

where $\chi \in C_0^\infty$ equals 1 in a neighorhood of K. Then

$$\mu(f) = -\frac{1}{\pi} \iint \frac{\partial \chi}{\partial \overline{\zeta}}(\zeta) \frac{f(\zeta)}{\zeta - z} \varphi(\zeta)\, d\lambda(s),$$

where

$$\varphi(\zeta) = \int_K \frac{1}{\zeta - z}\, d\nu(z) \in \mathrm{Hol}(\mathbb{C} \setminus K),$$

and at ∞,

$$\varphi(\zeta) = \sum \frac{1}{\zeta^{j+1}} \underbrace{\left(\int z^j\, d\nu(z) \right)}_{=\mu(z^j)} = B(\zeta).$$

So it is natural to look for this kind of representation of an analytic functional.

1.2 Universal covering spaces

Theorem 1.2. *Let X be a connected topological manifold. Then there exists a simply connected manifold $\tilde{X}$ and a covering map $p : \tilde{X} \to X$.*

Remark 1.2. If $\tilde{p} : \tilde{X} \to X$ and $\hat{p} : \hat{X} \to X$ are covering maps and $\tilde{X}, \hat{X}$ are simply connected, then there is a homeomorphism $f : \tilde{X} \to \hat{X}$ such that $\hat{p} \circ f = \tilde{p}$.

Proof. Let $x_0 \in C$, and let $\pi(x_0, x)$ be the set of homotopy classes of paths from x_0 to x. Define $\tilde{X} = \{(x, \Gamma) : x \in X, \Gamma \in \pi(x_0, x)\}$. Define the following topology on $\tilde{X}$: Let $(x, \Gamma) \in \tilde{X}$, and let U be a path-connected and simply connected neighborhood of X. Define $\langle U, \Gamma \rangle = \{(y, \Gamma) : y \in U, \Lambda = [\gamma * \alpha], \Gamma = [\gamma], \alpha \text{ from } x \text{ to } y\}$. Use the sets $\langle U, \Gamma \rangle$ as a base for a topology on $\tilde{X}$.

Let $p : \tilde{X} \to X$ send $(x, \Gamma) \mapsto x$. We claim that p is a covering map. Let $x \in X$, and let U be a path-connected and simply connected neighborhood of x. Then

$$p^{-1}(U) = \bigcup_{p(x,[\sigma])=x} \langle U, [\sigma] \rangle \,,$$

where σ is a path from x_0 to x. If $[\sigma] \neq [\tau]$, then $\langle U, [\sigma] \rangle \neq \langle U, [\tau] \rangle$: if $(y, [\gamma]) \in \langle U, [\sigma] \rangle \cap \langle U, [\tau] \rangle$, then there are paths α, β in U from x to y such that $[\gamma] = [\sigma * \alpha] = [\tau * \beta]$; α and β are homotopic, so $[\sigma] = [\tau]$.

One checks that p is continuous and open. Let us see that $p : \langle U, [\sigma] \rangle \to U$ is bijective:

- surjective: U is path-connected. p is injective:
- injective: Suppose $(y, [\tau]) = p(y, [\gamma])$. Then there are paths α, β from x to y such that $[\tau] = [\sigma * \alpha]$ and $[\gamma] = [\sigma * \beta]$. α and β are homotopic, so $[\tau] = [\gamma]$.

We have checked that $p : \tilde{X} \to X$ is a covering map.

It remains to show that $\tilde{X}$ is simply connected. We will do this next time. $\square$